

Massive sphere determinants

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An expression for the functional determinant on a sphere for a massive (scalar) field derived by Denef, Hartnoll and Sachdev using quasinormal modes is shown to exist already in the literature together with the multiplicative anomaly interpretation. The relevant expressions are outlined and several equivalent versions are given. The variation with mass is determined numerically.

As an application of the derived formulae, the Hartle–Hawking probability of the Universe (via the dS/CFT correspondance) is recomputed. Agreement is found with Anninos, Denef and Harlow. The calculation is extended to all (odd) dimensions. I also compute the wave function which reveals an interesting feature.

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1. Introduction.

The paper of Denef, Hartnoll and Sachdev, [1], on the computation of functional determinants by quasinormal modes contains, as a simple, illustrative example, that for a massive scalar field on a Wick rotated de Sitter space aka a sphere.

Although this is only one aspect of their paper, and further relevant work has appeared since, I wish to point out, somewhat belatedly, that their result (eqn.(84)) essentially occurs in [2]. This paper was concerned with the effective action on spherical factors computed by a direct spectral approach. Further analysis of the full sphere was given in [3] where a general formula for minimal coupling is given. The multiplicative anomaly interpretation also occurs in [2].

It might be useful to expose the details of this connection. In doing so, accelerated treatments of points raised in [1] are encountered. I also give some other representations of the determinant and present numerical results.

Incidentally, the general method of regularising factorised products, as employed in [1] was also used by Quine and Choi, [4], for spheres. A related technique is in the important paper by Voros, [5]. The connection of the method in [2] with canonical products was given in [6].

As an application of the derived formulae, I recalculate the probability of the universe from the Hartle–Hawking wave–function which the dS/CFT correspondence says is the determinant of the propagation operator, [7]. I do this for all odd dimensional spheres.

2. The essentials

A central tactic in [2] was to take the full sphere as the union of the Dirichlet and Neumann problems on the hemisphere, the basic calculational device being the ζ –function constructed from the relevant sets of eigenvalues. Then I write, $\zeta(s) = \zeta_N(s) + \zeta_D(s)$ for the full sphere.

Not only does this have analytical advantages but it gives access to the individual N and D quantities which have relevance for AdS.

For flexibility, I defined the object,

$$\zeta(s, a, \alpha \mid \boldsymbol{\omega}) = \sum_{\mathbf{m}=0}^{\infty} \frac{1}{((a + \mathbf{m} \cdot \boldsymbol{\omega})^2 - \alpha^2)^s}, \quad (1)$$

where \mathbf{m} and $\boldsymbol{\omega}$ are d –vectors, so that, on the full sphere, S^d ,

$$\zeta(s) = \zeta(s, a_N, \alpha \mid \mathbf{1}) + \zeta(s, a_D, \alpha \mid \mathbf{1}), \quad (2)$$

where $a_N = (d-1)/2$, $a_D = a_N + 1$ and where $\mathbf{1}$ is a d -vector. The real numbers ω are the *parameters*. I could keep them general. However, for ease, I have immediately selected the hemisphere by setting them all to unity and they are henceforth dropped.

If the field is conformal in $(d+1)$ dimensions, then $\alpha = 0$, and $\zeta(s, a, \alpha)$ reduces to the more elegant Barnes function. For conformal in d -dimensions $\alpha = 1/2$ while for minimal coupling $\alpha = (d-1)/2$. These were the only values considered for computation in the cited earlier references but in [8] I extended some evaluations numerically to the massive field by taking $\alpha^2 = 1/4 - \mu^2$. The analysis in [2] remains valid in this case and it is this that I wish to outline here.

In these evaluations of the required quantity, $\zeta'(0, a, \alpha)$, an expansion in α sufficed and allowed the Barnes function, ζ_d , to be brought in as the zeroth order term. Since the algebra is exposed in the references I need quote only the answer,

$$\zeta'(0, a, \alpha) = \zeta'_d(0, a + \alpha) + \zeta'_d(0, a - \alpha) + M(a, \alpha) \quad (3)$$

where the multiplicative anomaly, $M(a, \alpha)$, [2], is given by the polynomial,

$$M(a, \alpha) = - \sum_{r=1}^{[d/2]} \alpha^{2r} \frac{H_{r-1}^O}{r} N_{2r}(d, a), \quad (4)$$

with H_r^O the odd harmonic number,

$$H_r^O = \sum_{k=0}^r \frac{1}{2k+1}.$$

The N_l are the residues of the Barnes function,

$$\zeta_d(s+l, a) \rightarrow \frac{N_l(d, a)}{s} + R_l, \quad \text{as } s \rightarrow 0, \quad (5)$$

and are given by generalised Bernoulli polynomials,

$$N_l(d, a) = \frac{1}{(l-1)!(d-l)!} B_{d-l}^{(d)}(a),$$

which are easily computed, therefore so is the multiplicative anomaly.

From a parity property of the Bernoulli polynomials, the multiplicative anomalies for Neumann and Dirichlet conditions are equal (opposite) for even (odd) d . So in the odd case they cancel on addition of the N and D quantities when finding the full sphere quantities.

The definition of mass in [1] is as deviation from minimal coupling, whereas mine, *e.g.* [8,9], is deviation from conformal. In any case, the parameter $i\nu$ in [1] is the same as my α , which aids comparison.

3. Actual evaluations

Equation (2) with (3) could be taken as the final answer but actual evaluation devolves, as usual, mostly upon the computation of the derivatives of the Barnes ζ -function.

When all the parameters are unity, the traditional way of doing this uses Barnes' expansion in terms of Hurwitz ζ -functions, [10], which results from the expansion of the degeneracy,

$$\zeta_d(s, a) = \sum_{r=1}^d \frac{(-1)^{d-r}}{(r-1)!(d-r)!} B_{d-r}^{(d)}(a) \zeta_R(s+1-r, a). \quad (6)$$

An example is always helpful, as in [1]. For the two-sphere, (6) reads,

$$\begin{aligned} \zeta_2(s, a) &= B_0^{(2)}(a) \zeta_R(s-1, a) - B_1^{(2)}(a) \zeta_R(s, a) \\ &= \zeta_R(s-1, a) - (a-1) \zeta_R(s, a). \end{aligned}$$

From (3) and (2) I need to set $a = a_N = 1/2$ and $a = a_D = 3/2$ in turn. Since $a_D = a_N + 1$, I can turn the a_D terms into a_N ones using the rearrangement,

$$\zeta_R(s, a+1) = \zeta_R(s, a) - a^{-s}$$

Therefore adding the N and D contributions, (2), I will require,

$$\begin{aligned} &\zeta_2(s, a) + \zeta_2(s, a+1) \\ &= \zeta_R(s-1, a) - (a-1) \zeta_R(s, a) + \zeta_R(s-1, a+1) - a \zeta_R(s, a+1) \\ &= \zeta_R(s-1, a) - (a-1) \zeta_R(s, a) + \zeta_R(s-1, a) - a^{-s+1} \\ &\quad - a \zeta_R(s, a) + a^{-s+1} \\ &= 2\zeta_R(s-1, a) - (2a-1) \zeta_R(s, a). \end{aligned} \quad (7)$$

A neater manipulation is contained in section 5.

The combination appearing in (3) can now be effected, by setting $a = a_N + \alpha, \equiv \Delta_+$, and $a = a_N - \alpha, \equiv \Delta_-$, in (7)² in turn and adding. This yields for the derivative at zero of (2), our answer,³

$$\zeta'(0) = \sum_{\pm} (2\zeta'_R(-1, \Delta_{\pm}) - (2\Delta_{\pm} - 1)\zeta'_R(0, \Delta_{\pm})) + 2M(1/2, \alpha), \quad (8)$$

which compares exactly with [1] equn. (79) on noting that the multiplicative anomaly is, from (4),

$$2M(1/2, \alpha) = -2\alpha^2 N_2(2, 1/2) = -2\alpha^2.$$

² For comparison purposes, I have introduced the notation of [1].

³ I sometimes refer to this quantity as 'minus logdet'.

4. The log term

I consider (8) as, to a factor, the finite part of the effective action. The divergence, and hence the associated log term, seen in [1], is driven by the value of the ζ -function at 0, $\zeta(0)$. This is somewhat easier to evaluate, [2], equn (22), and is

$$\zeta(0) = \frac{1}{2}(\zeta_d(0, a + \alpha) + \zeta_d(0, a - \alpha)) \quad (9)$$

summed for N and D conditions. (For conformal couplings this would be the conformal anomaly on the sphere.)

Then, again using (7),

$$\begin{aligned} \zeta(0) &= \sum_{\pm} \zeta_R(-1, \Delta_{\pm}) - (\Delta_{\pm} - 1/2)\zeta_R(0, \Delta_{\pm}) \\ &= -\frac{1}{2} B_2(\Delta_{\pm}) + (\Delta_{\pm} - 1/2)^2 \\ &= -\frac{1}{6} + \alpha^2 \end{aligned} \quad (10)$$

to be compared with [1].

The value of the Barnes ζ -function, $\zeta_d(0, a)$, is a polynomial in a and, combining the log terms with the multiplicative anomaly, gives the quantity denoted by $\text{Pol}(\nu)$ in [1].

5. General dimensions

A formula like (8) plainly holds for any sphere. I outline a few details making everything explicit, but I will not combine all the pieces.

Adding the N and D parts entails the combination,

$$\zeta_d(s, a + 1) + \zeta_d(s, a) = 2\zeta_d(s, a) - \zeta_{d-1}(s, a),$$

where I have used the basic recursion for the Barnes ζ -function. If (6) is substituted into the right-hand side one gets the explicit expression,

$$\frac{2}{(d-1)!} \zeta_R(s+1-d, a) + \sum_{r=1}^{d-1} \frac{(-1)^{d-r}}{(r-1)!} \left(\frac{2B_{d-r}^{(d)}(a)}{(d-r)!} - \frac{B_{d-1-r}^{(d-1)}(a)}{(d-1-r)!} \right) \zeta_R(s+1-r, a)$$

whose derivative at zero can be taken and the sum on $a = \Delta_{\pm}$ performed.

The coefficient of the log terms follows easily from (9) using the standard formula,

$$\zeta_d(0, a \pm \alpha) = \frac{(-1)^d}{d!} B_d^{(d)}(a \pm \alpha) = \frac{1}{d!} B_d^{(d)}(d - a \mp \alpha),$$

the sum of which is a polynomial in α^2 and generalises (10).

Also, noting the duality, $d - (a_D \pm \alpha) = a_N \mp \alpha$ the Dirichlet value is equal or opposite to the Neumann, depending on whether d is even or odd, just as for the multiplicative anomaly. Hence the full sphere values are zero in odd dimensions, as is well known and must be so for manifolds without boundary, on general grounds. Multiplied by the scaling log and added to the multiplicative anomaly, this is $\text{Pol}(\nu)$ of [1]. As has just been shown, this is zero in odd dimensions (on the full sphere).

Putting all these ingredients together, it is clear that the final answer for $\zeta'(0)$ is of the form of eqn.(84) in [1]. I will not write it out. A similar, general expression for minimal coupling was given in [3] where a graph was drawn. See also [4].

6. Large mass

According to standard theory, the large mass expansion of the effective action is locally determined by heat-kernel coefficients. This entered into our considerations in [8] the point being that for a propagation equation that is conformal in $d + 1$ dimensions, the heat-kernel *terminates* for odd spheres, an old fact. The coefficients are computed explicitly in [11] and the expansion is written out in [8]. It turns out to be one in ν so that, for all odd dimensions, up to exponential corrections,

$$\zeta'(0) \sim 2\pi \sum_{k=1,3,\dots}^d \frac{(-1)^{(k+1)/2}}{k!(d-k)!} B_{d-k}^{(d)}((d-1)/2) \nu^k, \quad (11)$$

in terms of generalised Bernoulli polynomials computation of which is easy. Four examples are,

$$\begin{aligned} \zeta'(0)|_{d=3} &\sim \frac{\pi}{3} \nu^3 \\ \zeta'(0)|_{d=5} &\sim -\frac{\pi}{12} \nu^3 \left(\frac{1}{5} \nu^2 + \frac{1}{3} \right) \\ \zeta'(0)|_{d=7} &\sim \frac{\pi}{90} \nu^3 \left(\frac{1}{28} \nu^4 + \frac{1}{4} \nu^2 + \frac{1}{3} \right) \\ \zeta'(0)|_{d=11} &\sim \frac{\pi}{9450} \nu^3 \left(\frac{1}{2112} \nu^8 + \frac{5}{288} \nu^6 + \frac{13}{64} \nu^4 + \frac{41}{48} \nu^2 + 1 \right), \end{aligned} \quad (12)$$

the first three being given in [1].⁴ The leading term equals $(-1)^{(d+1)/2} (2\pi/d!) \nu^d$ which is just the Weyl universal value.

The asymptotic behaviour for even dimensions is a little different, and was given early on, [12] equ.(17). It is again determined by the heat-kernel coefficients but I give just the leading term,

$$\zeta'(0) \sim (-1)^{d/2} \frac{|S^d|}{(4\pi)^{d/2}} (H_{d/2} - \log \nu^2) \nu^d, \quad \nu \rightarrow \infty, \quad (13)$$

where H_n is the harmonic number, $H_n = \sum_{k=1}^n 1/k$.

7. Alternative expressions. Numerical evaluation

The expression for the logdet, or $\zeta'(0)$, in terms of the derivatives of the Hurwitz ζ -function can be used to calculate numerical values perfectly reasonably, in most languages. However in this section, I first give another form for $\zeta'(0)$ which has formal, and practical implications. I shall mostly be repeating known things, but it is handy to have them visible.

The multiple Γ -function is defined in terms of the derivative at 0 of the Barnes ζ -function so that, essentially as a definition, (3) can be rewritten (I drop the **1** as understood),

$$\zeta'(0, a, \alpha) = \log \frac{\Gamma_d(a + \alpha) \Gamma_d(a - \alpha)}{\rho_d^2} + M(a, \alpha)$$

where $\rho_d = \Gamma_{d+1}(1)$ is the multiple modular form. The recursion for Γ_d allows this to be transformed into,

$$\zeta'(0, a, \alpha) = \log \frac{\Gamma_{d+1}(a + \alpha) \Gamma_{d+1}(a - \alpha)}{\Gamma_{d+1}(a + \alpha + 1) \Gamma_{d+1}(a - \alpha + 1)} + M(a, \alpha)$$

Adding the N and D expressions, and remembering that $a_D = a_N + 1$ yields, on the full sphere therefore,

$$\zeta'_{FS}(0, \alpha) = \log \frac{\Gamma_{d+1}(a + \alpha) \Gamma_{d+1}(a - \alpha)}{\Gamma_{d+1}(a + \alpha + 2) \Gamma_{d+1}(a - \alpha + 2)} + 2M(a, \alpha) \delta_{d, \text{even}} \quad (14)$$

where here $a = a_N = (d - 1)/2$.

⁴ There seems to be a misprint in [1] for $d = 7$.

As usual, for odd d things simplify. For a start, there is no multiplicative anomaly. Then the remaining Γ -part can be written in terms of multiple trig functions, and, ultimately in terms of ordinary functions. Thus, *cf* [13],

$$\begin{aligned}\zeta'_{FS}(0, \alpha)|_{odd} &= \log \text{Sin}_{d+1}(a - \alpha) - \log \text{Sin}_{d+1}(a - \alpha + 2) \\ &= - \int_{a-\alpha}^{a-\alpha+2} dz \text{Cot}_{d+1}(z) \\ &= \frac{1}{d!} \int_{a-\alpha}^{a-\alpha+2} dz B_d^{(d+1)}(z) \pi \cot(\pi z)\end{aligned}\tag{15}$$

which is suitable for numerical treatment.⁵

Depending on the range of the mass, α can be imaginary, and then the real part of the integral should be taken. This follows from the symmetry in α , (not apparent in the form (15)).

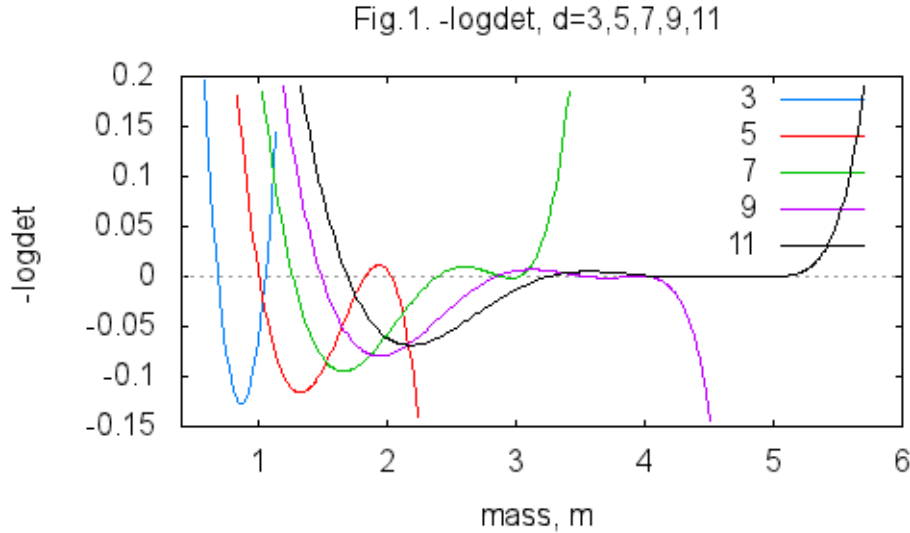


Figure 1 shows the values of $-\log\text{det}$ for the first five (odd) dimensions. Small mass shows the zero mode logarithmic divergence. For increasing m there is a minimum and then the values oscillate about zero (with decreasing amplitude as d becomes larger) before asymptotic behaviour sets in. For very big dimensions the graph is approximately zero, except for the two asymptotic regions.

This technique works only for odd dimensions. Another approach, valid for both odd and even d , consists of integrating the multiple digamma function, ψ_d ,

⁵ The Bernoulli polynomial is actually a forwards factorial and the formula is Plancherel looking. For example, in $d = 3$, $B_3^{(4)}(x) = (x - 1)(x - 2)(x - 3)$.

defined by,

$$\psi_d(z) = \frac{\partial}{\partial z} \log \Gamma_d(z),$$

because ψ_d is expressible in terms of the ordinary digamma function which is available numerically. This results in the combination appearing in (14),

$$\log \frac{\Gamma_d(z_2)}{\Gamma_d(z_1)} = \int_{z_1}^{z_2} dz \psi_d(z), \quad (16)$$

with

$$\psi_d(z) = \frac{(-1)^{d-1}}{(d-1)!} \left(B_{d-1}^{(d)}(z) \psi(z) + Q_d(z) \right), \quad (17)$$

where the polynomial Q is given by,

$$Q_d(z) = -(-1)^{d-1} \sum_{n=1}^{d-1} \frac{(-1)^n}{n} B_{d-n-1}^{(d-n)}(d-z) B_n^{(n)}(z).$$

The expression for ψ_d , (17), follows, [13], on iteration of recursion relations given by Barnes, [14]. (See also Onodera, [15].) For example,

$$\psi_4(z) = -\frac{z(z-1)(z-2)}{6} \psi(z) + \frac{22z^3 - 114z^2 + 167z - 60}{72}.$$

Dropping the multiplicative anomaly term for the moment, (14) gives

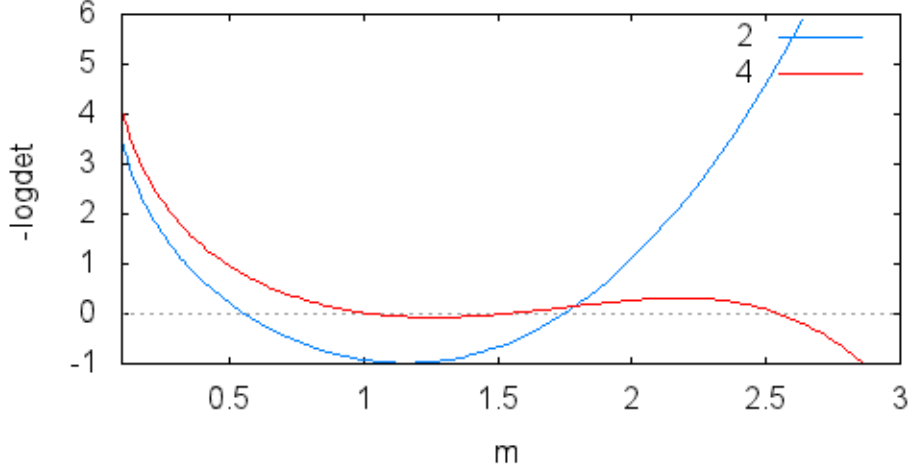
$$\begin{aligned} \zeta'_{FS}(0, \alpha) &= \left(\int_{a+\alpha}^{a+2+\alpha} + \int_{a-\alpha}^{a+2-\alpha} \right) dz \psi_{d+1}(z) \\ &= \int_a^{a+2} dz (\psi_{d+1}(z+\alpha) + \psi_{d+1}(z-\alpha)), \end{aligned} \quad (18)$$

which can be calculated by substituting (17) and using numerics for the complex ψ -function. For even dimensions, the explicit multiplicative anomaly, (4), has to be added to give the total $-\log\det$.

As a check of the numerical procedure, agreement with the known values for the logdet for conformal coupling (in d -dimensions) is found. Additional verification is provided by the leading asymptotic behaviour, (13), which, for example, is valid to 7 figures for $\nu \sim 1000$.

Figure 2 shows the values for $d = 2$ and $d = 4$.

Fig.2 -logdet, d=2,4



Another method, that works only for odd dimensions, was employed in [8] for a similar massive purpose (see also [9]) and is based on a continuation given by Minakshisundaram, [16], and employed by Candelas and Weinberg, [17], and by Chodos and Myers, [18].

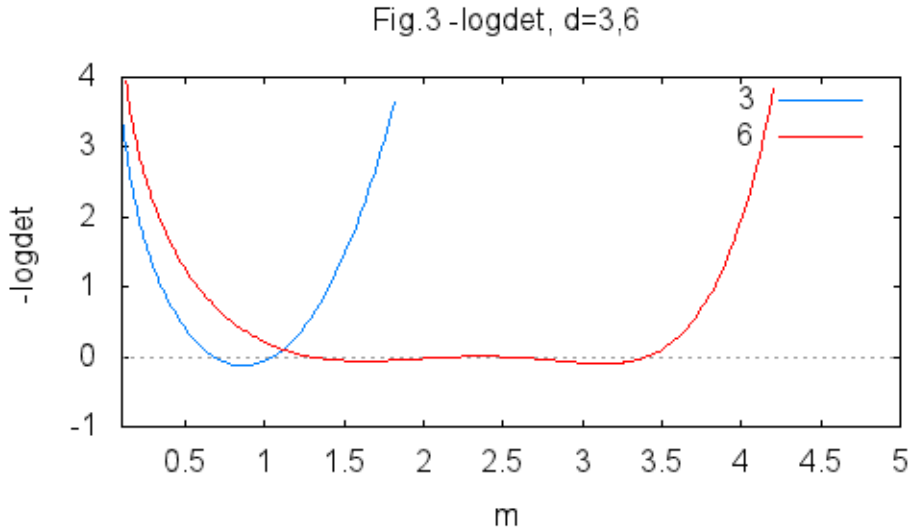
Since the details of the continuation are thus readily available, I describe only the basic idea which is to employ a Bessel transform for $(\lambda - \alpha^2)^{-s}$ that separates λ and α . In the present case of (1), this again has the effect of introducing Barnes spectral quantities and allows the sums over \mathbf{m} to be done, introducing thereby the heat-kernel for the pseudo-operator with eigenvalues $\sqrt{\lambda}$.⁶ In the present case these are either integers or half-integers.

In order for the method to proceed as in the cited references without modification, it is necessary to deal with the full sphere expression. The result of the continuation to $s = 0$ then gives, [8],

$$\begin{aligned}\zeta'_{FS}(0) &= 2 \int_{-\infty+iy}^{\infty+iy} \frac{d\tau \cosh \tau/2 \cosh \alpha\tau}{\tau \prod_{i=1}^d 2 \sinh(\tau/2)} \\ &= \frac{1}{2^{d-2}} \int_0^\infty dx \operatorname{Re} \frac{\cosh \tau/2 \cosh \alpha\tau}{\tau \sinh^d(\tau/2)}, \quad \tau = x + iy,\end{aligned}\tag{19}$$

where y lies between 0 and 2π . This is easily computable and produces values that agree very precisely with those from (15) or (18), which is comforting. The general behaviour is the same as for even d . Figure 3 compares $d = 3$ and $d = 6$.

⁶ This kernel also goes by the names of cylinder kernel, single-particle partition function or generating function, [11].



8. An application. The wave function of the universe.

Interest is attached to the behaviour of the determinant as an analytic function of a complex mass, in particular for negative mass squared. One conjecture in the dS/CFT dictionary is that the Hartle–Hawking wavefunction in the bulk is equal to the partition function (determinant) of a CFT. More precisely, defining mass as deviation from conformal, the relevant parameter, σ , is defined by $\alpha^2 = 1/4 - \sigma$. For a de Sitter bulk, specific computations have been performed by Anninos, Denef and Harlow, [7] and Anninos *et al*, [19] interpreting σ as a (uniform) massive deformation. To illustrate the application of the expressions here, I will reobtain their results and extend them to any odd dimension.

The formula I take for continuation is (15). (I will use α for the analysis.) Since α can now become larger than a , the denominator in the ζ -function, (2) with (1), can become zero, and then negative. This introduces an imaginary part to $\log\det$ and, therefore, a phase (a sign) to the determinant, as is well known. This will come about from the poles of the cotangent, on my definition, and, as a continuation, I take the integration interval as a complex path say just above, or just below the real axis with end points at $z = a - \alpha$ and $z = a + 2 - \alpha$ which slide along the real z -axis as α varies. These paths, and any others that might be chosen, give the same result for the determinant as I now explain, in words.

Because of the factorial nature of the Bernoulli polynomial (it has zeros) the first cotangent pole to worry about is always at $z = 0$. This first comes into play

when $\alpha = a$, as expected, giving a (single) zero eigenvalue. At this point the sign of the resulting infinity is such that the determinant vanishes. As α increases beyond a the next zero eigenvalue (giving infinity) occurs at $\alpha = a + 1$ and corresponds to the pole at $z = -1$. The attached zero has a higher degeneracy of $(d + 1)$. Thereafter, infinities occur at $\alpha = a + n$ ($n = 2, 3, \dots$), with the offending cotangent poles now being at *both* ends of the integration range *i.e.* at $z = -n + 2$ and $z = -n$.

For an α *not* giving a zero mode (leading to an infinity), apart from the first instance there will be two cotangent poles at the integers $-n$ and $-n + 1$, say, just above or just below the contours. Now move these contours into coincidence. The residues at the poles are integers, as follows from (15), and so the values of logdet differ by integer multiples of 2π thus rendering the determinant unambiguous, as stated. The contour can now be run along the real axis with indentations at the two (or one) cotangent poles. The real part is the principle part, which has to be calculated numerically. The imaginary part equals $i\pi N$ with N an integer given by the residue,

$$N = \frac{1}{d!} (B_d^{(d+1)}(-n) + B_d^{(d+1)}(-n + 1)).$$

Without entering into motivation or explanation, [7,19], the quantity of interest is $|\Psi_{HH}|^2 = |Z|^2$ where Z is the determinant of the propagating operator, in this case the one on the sphere that gives the eigenvalues occurring in (1), (2). By definition, Z is automatically finite here and given by $Z = e^{-\zeta'_{FS}(0,\alpha)}$. The phase of Z is irrelevant in computing $|Z|^2$ and so one needs just the principle part of the integral in (15) which is easily found.

Figure 4 shows the variation of $|Z|^2$ plotted ⁷ against σ to aid comparison with [7], [19] with which the $d = 3$ case agrees. The $d = 5$ values are new and a difference is that, this time, there is a *minimum* at the conformal point $\sigma = 0$. This behaviour alternates with dimension and I plot the $d = 7$ values in Figure 5.

The increasing broadness of the zeros as σ decreases is a consequence of the increasing degeneracy of the relevant zero. This can be made more precise, but I won't bother.

For comparison I have plotted, in Figure 6, the wave function itself, $Z \sim \Psi_{HH}$ with the signs taken into account. The results seem to show that for $d = 3$ the zero at $\sigma = -.75$ for the probability is actually a cusp, and the same for $d = 5$ at $\sigma = -3.75$.

⁷ I have not given the positive σ values as they were discussed above.

Fig.4. Probability density of the Universe, $d=3,5$

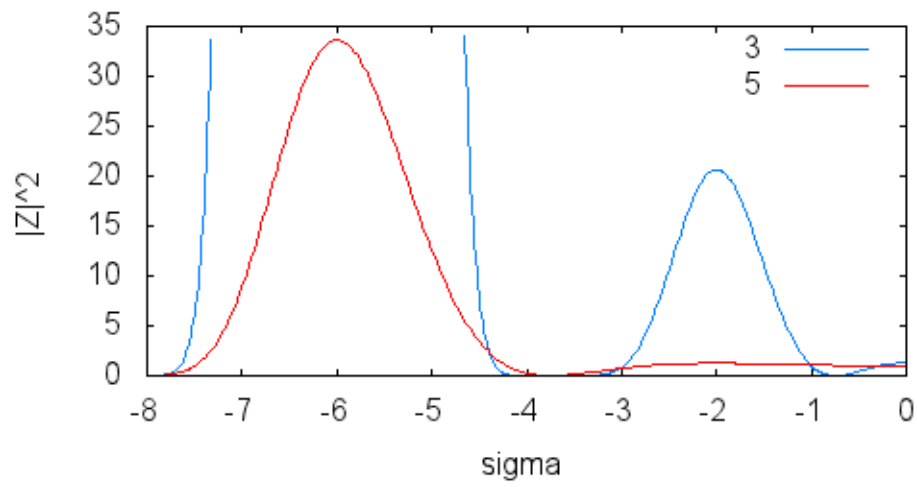


Fig.5. Probability density of the Universe, $d=7$

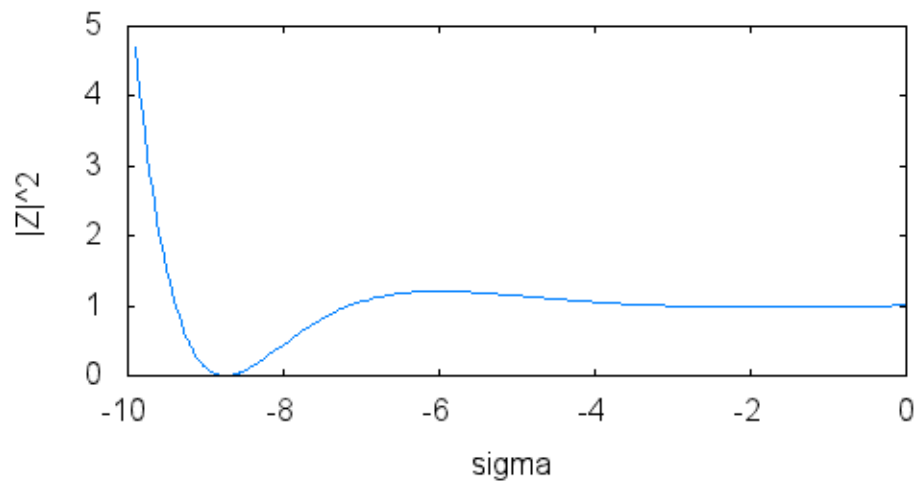
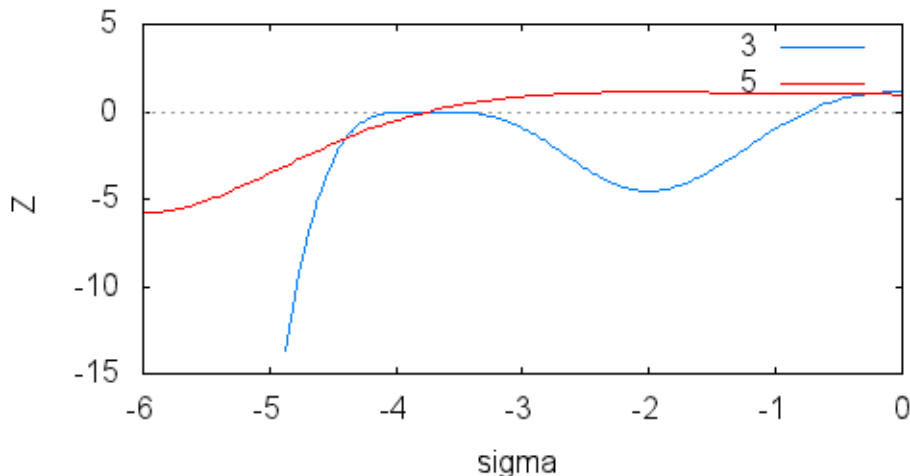


Fig.6. Wave function of the Universe, $d=3,5$



10. Discussion

I have given an alternative treatment of the functional determinants for a massive field on spheres. Agreement is found with existing results, which are extended to arbitrary dimensions.

It is not necessary in the present method to know the heat-kernel coefficients explicitly in order to find the polynomial. They are encoded in the values of the ζ -function.

Spreafico, [20]⁸, gives a calculation for the massive two-sphere that involves an infinite product for the multiplicative anomaly. As also noted in [1], he does not include the massive deformation of the minimal zero mode in the ζ -function. It is included here because I have *defined* logdet to be $-\zeta'(0)$. This gives the usual value for the determinant in the conformal case, for example, since there is now no *mathematical* reason for excluding the lowest mode. *Physically*, for the effective action, one might absorb its contribution into any scaling log terms for renormalisation.

Quotients of the sphere, such as lens spaces, can be treated without difficulty by reinstating the parameters, ω , [2,9,21].

⁸ Incidentally, regarding a comment in this reference, the value for the minimal $\zeta(0)$ on the two-sphere in [2] is actually correct since $\zeta(0) = 2\zeta_D(0) - 1 = 2 \times 1/6 - 1 = -2/3$. Also I find myself unable to reproduce some of the values on the projective spaces in [20].

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